Global Optimization of a Nonconvex Single Facility Location Problem by Sequential Unconstrained Convex Minimization*

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Abstract. The problem of maximizing the sum of certain composite functions, where each term is the composition of a convex decreasing function, bounded from below, with a convex function having compact level sets arises in certain single facility location problems with gauge distance functions. We show that this problem is equivalent to a convex maximization problem over a compact convex set and develop a specialized polyhedral annexation procedure to find a global solution for the case when the inside function is a polyhedral norm. As the problem was solved recently only for local solutions, this paper offers an algorithm for finding a global solution. Implementation and testing are not treated in this short communication.

Key words. Facility location, polyhedral annexation method.

1. Introduction

In a recent paper, Idrissi *et al.* [5] consider the following class of global optimization problems (for the case $m = 2$)

$$\max_{x \in R^m} \sum_{j=1}^{n} q_j[h_j(x)] \tag{P}$$

under the assumptions that: (A1) q_j is a strictly convex function, strictly decreasing on R_+, with values in R_+, for which $\lim_{t\to\infty} q_j(t) = 0$ for all j; and (A2) h_j is a convex function defined on R^m, with values in R_+, such that $\lim_{|x|\to\infty} h_j(x) = +\infty$ for all j, where $|\cdot|$ denotes an arbitrary norm.

The objective function $\varphi(x) := \Sigma_{j=1}^{n} q_j[h_j(x)]$ is generally neither convex nor concave. In [5], a method that computes only a local (instead of global) maximum is proposed. The aim of this paper is to present a method for finding a *global* maximum. The method reduces to solving a sequence of unconstrained convex minimization problems of the form

$$\min_{x \in R^m} \sum_{j=1}^{n} \alpha_j h_j(x)$$

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for $\alpha_j \geq 0$. The method is a specialization of the polyhedral annexation procedure earlier developed in [8] (see also [4]). The utilization of the general procedure herein demonstrates the value and versatility of polyhedral annexation approaches.

As noted in [5], for the case $m = 2$, problem (P) arises in certain single facility location problems. Specifically, consider the problem of locating a "desirable" facility (such as community center or branch library) which is designed to serve n districts of a town or region. District j has a population located at $a_j \in R^2$, $j = 1, \ldots, n$, which can be interpreted as the population center of mass. Let $x \in R^2$ denote the location of the desirable facility and take $h_j(x) = |x - a_j|$ as the distance function from x to district j using an arbitrary polyhedral norm $|\cdot|$ (see, e.g., [7]). Then $q_j(|x - a_j|)$ measures the attraction of the district j population to the desirable facility when it is located at x. According to assumption (A1), the farther x is away from a_j the less attractive it looks to the district j population. Assumption (A2) formalizes the property that customers from district j must travel an infinite distance to a new facility that is located infinitely far away from the origin. Thus, problem (P) is to determine the optimal location $x \in R^2$ that maximizes the total attraction.

In general, a different norm can be associated with each district. When the distance measures between points are not symmetric (this occurs when there are one-way streets, for example) then gauges can be used for h_j instead of norms.

Problem (P) is more difficult than the familiar Fermat–Weber problem with nonlinear costs [2] where the objective is to *minimize* and the "cost" functions $q_j: R_+ \to R_+$ are assumed to be strictly increasing and convex such that $q_j(0) = 0$. Note that the Fermat–Weber problem is to minimize a convex function of a distance measure while problem (P) *maximizes* a convex function of the same measure. In fact, we show in Section 2 that problem (P) is equivalent to a convex maximization problem.

Idrissi *et al.* [5] develop a procedure based on solving a sequence of parameterized Fermat–Weber problems for finding only a local solution of problem (P). As in the more general case treated in [2], all local solutions of problem (P) are among the *intersection points* of the polyhedral norm. These points can be very easily characterized as follows. Let $\{v^1, v^2, \ldots, v^{s_j}\}$ denote the vertices of the "unit ball" in R^2 associated with the polyhedral norm of district j. For each a_j, $j = 1, \ldots, n$, draw the s_j half-lines emanating from a_j through v^i for all $i = 1, \ldots, s_j$. An intersection point is where two noncollinear half lines, emanating from a_k and a_l $(k, l = 1, \ldots, n)$, cross. It is easy to see that there are finitely many intersection points and that R^2 is thus partitioned into finitely many polyhedra each having vertices defined by intersection points and set directions defined by the half-lines constructed above. The polyhedra are called *elementary polyhedra* [7] (see Figure 1). It is demonstrated in [5] that the objective function of (P) is convex on every elementary polyhedron; thus, the problem can be viewed as that of maximizing a piecewise convex function when R^2 is tiled by

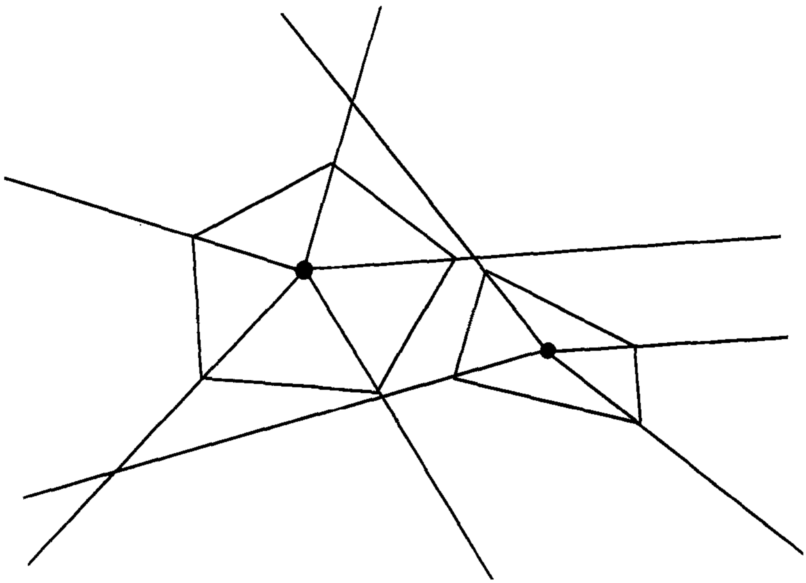

Fig. 1. Intersection points of polyhedral norms and elementary polyhedra.

elementary polyhedra. The algorithm in [5] effectively uses a parameterized partial linearization of the objective to achieve linearity over each elementary polyhedron which gives rise to a Fermat–Weber problem that is solved for successively updated settings of a fixed parameter vector. Only improving vertices of elementary polyhedra are generated until a local solution is encountered.

In contrast, our general approach is based on our observation in Section 2 that problem (P) can be reformulated as a constrained convex maximization problem which is a "classical" problem in global optimization. Consequently, our approach does not break down for $m \geq 3$ as does the algorithm in [5]. Also, we simply assume the convexity instead of strict convexity of $q_j(t)$. We employ an existing algorithm based on successively enlarging polyhedra and specialize this procedure in Section 3 to the problem at hand. In Section 4 we prove that every limit point of our algorithm is a global solution to problem (P). The closing Section 5 discusses some implementation considerations and techniques for computing certain quantities required by the algorithm. The treatment in the remainder of the paper assumes a basic knowledge of the fundamental techniques in deterministic global optimization, as detailed in [4].

2. Basic Properties

In this section we first prove that problem (P) is equivalent to a constrained convex maximization problem that can be solved by an existing polyhedral annexation procedure. When this procedure is specialized to our problem, we show that it reduces to solving a sequence of unconstrained convex minimization subproblems.

PROPOSITION 1 (see [5], Proposition 2.1). *Let* $\varphi(x) = \Sigma_{j=1}^{n} q_j[h_j(x)]$. *Then* $\varphi(x)$ *has a global maximizer on* R^m. *In particular the function* $\varphi(x)$ *is bounded on* R^m.

PROPOSITION 2. *If* $\bar{x}$ *solves* (P) *then* $(\bar{x}, \bar{t})$ *with* $\bar{t} = h(\bar{x})$ *solves the convex maximization problem*

$$\max_{(x,t)}\left\{\sum_{j=1}^{n} q_j(t_j) : h_j(x) \leq t_j (j = 1, \ldots, n), (x, t) \in R^{m+n}\right\}. \tag{Q}$$

Conversely, if $(\bar{x}, \bar{t})$ *solves* (Q) *then* $\bar{x}$ *solves* (P).

Proof. If (x, t) satisfies $h(x) \leq t$ and $\Sigma_{j=1}^{n} q_j(t_j) > \Sigma_{j=1}^{n} q_j[h_j(\bar{x})]$ then $\Sigma_{j=1}^{n} q_j[h_j(x)] \geq \Sigma_{j=1}^{n} q_i(t_j) > \Sigma_{j=1}^{n} q_j[h_j(\bar{x})]$. This proves the first assertion. To prove the second assertion, observe that if $\Sigma_{j=1}^{n} q_j[h_j(x)] > \Sigma_{j=1}^{n} q_j[h_j(\bar{x})]$ then for $t = h(x)$ we would have

$$\sum_{j=1}^{n} q_j(t_j) = \sum_{j=1}^{n} q_j[h_j(x)] > \sum_{j=1}^{n} q_j[h_j(\bar{x})] \geq \sum_{j=1}^{n} q_j(\bar{t}_j) . \qquad \square$$

Thus the problem is a convex maximization problem which can be solved by a general purpose global optimization method (see [8]). We exploit the special structure of the problem to obtain a more efficient streamlined version of this procedure.

Denote

$$\tilde{D} = \{t \in R_+^n : h(x) \leq t \text{ for some } x \in R^m\}$$

$$f(t) = \sum_{j=1}^{n} q_j(t_j) , \tag{1}$$

where $h: R^m \to R_+^n$. Then problem (Q) can be rewritten as

$$\max_{t \in \tilde{D}} f(t) , \tag{$\tilde{Q}$}$$

Let $\bar{t}$ be the best feasible solution of $(\tilde{Q})$ known at a certain stage. The core of our method is a procedure for solving the following subproblem:

$(\tilde{Q}, \bar{t})$: *Determine whether* $f(t) \leq f(\bar{t})$ *for all* $t \in \tilde{D}$ *(i.e.,* $\bar{t}$ *is globally optimal) and if not, find a feasible solution* t' *such that* $f(t') > f(\bar{t})$.

Let $t^\circ \in \tilde{D}$ be such that $f(t^\circ) < f(\bar{t})$. Denote $\gamma = f(\bar{t})$, $\tilde{C} = \{t \in R^n : f(t) \leq \gamma\}$, $D = \tilde{D} - t^\circ$ and $C = \tilde{C} - t^\circ$. Then the set C is convex, closed, $0 \in D \cap \operatorname{int} C$ and $(\tilde{Q}, \bar{t})$ can be reformulated as:

$(\tilde{Q}, \bar{t})$: *Determine whether* $D \subset C$ *and if not find a point in the difference* $D \backslash C$.

For solving this problem, we observe the following properties of C and D.

PROPOSITION 3. *C contains the orthant R^n_+.*

Proof. For any $t \in R^n_+$ we have $f(t^\circ + t) \leq f(t^\circ)$ because each $q_j(\cdot)$ is strictly decreasing on R_+. Hence $f(t^\circ + t) \leq \gamma$; i.e., $t^\circ + t \in \tilde{C}$ or $t \in C$. □

PROPOSITION 4. *For any vector $s \in R^n_+$ we have*

$$\min_{t \in \tilde{D}} \sum_{j=1}^{n} s_j t_j = \min_{x \in R^m} \sum_{j=1}^{n} s_j h_j(x) .$$

Proof. Obvious from (1) and the fact $s \in R^n_+$. □

It turns out that, due to these properties, the polyhedral annexation procedure in [8] specialized to problem $(\tilde{Q}, \bar{t})$, will reduce to solving a sequence of unconstrained convex minimization subproblems. We show this in the next section. An example is presented first to illustrate the main constructions in our procedure.

EXAMPLE As in [5], consider the attraction function for district j given by $q_j(t_j) = \alpha_j \omega_j e^{-\beta_j t_j}$ where α_j is the average fraction of the population of size ω_j that frequents the center. Small values of the parameter $\beta_j \geq 1$ magnify the importance of district j's proximity to the center.

Let the distance from x to district j be $h_j(x) = |x - a_j|$, where $|\cdot|$ is any given norm. By Proposition 2, problem (P) for this example is equivalent to

$$\max_{(x,t)} \sum_{j=1}^{n} \alpha_j \omega_j e^{-\beta_j t_j}$$

$$\text{s.t. } |x - a_j| \leq t_j \quad (j = 1, \ldots, n) .$$

The variable x in this problem, as in problem (Q) in the general case, only plays an intermediate role. Therefore, it is possible and more efficient to solve it as a problem in the variables $t_j (j = 1, \ldots, n)$ only; i.e., as a problem in R^n rather than $R^m \times R^n$. In the next section we will show that this can be done via a dualization procedure.

3. Solution Method

Let C^* and D^* denote the polars of C and D, respectively. From general properties of polars, we have $D \subset C$ if and only if $C^* \subset D^*$. We now consider how to check $C^* \subset D^*$.

From Proposition 3 it follows that $C^* \subset R^n_-$, and from $0 \in \text{int}C$ it follows that C^* is compact (see, e.g., [6]).

Now consider a polytope S such that $C^* \subset S \subset R^n_-$. We are interested in knowing whether $S \subset D^*$, because if this holds then *a fortiori* $C^* \subset D^*$.

PROPOSITION 5. *Let V be the vertex set of S. We have $S \subset D^*$ if and only if*

$$\max\left\{\sum_{j=1}^{n} v_j t_j : t \in D\right\} \leq 1 \quad \textit{for all} \quad v \in V . \tag{2}$$

Proof. Since D^* is convex, we have $S \subset D^*$ if and only if $V \subset D^*$. But from the definition of polars, $v \in D^*$ if and only if (2) holds. □

Thus, to check whether or not $S \subset D^*$, we solve the subproblem $\max\{\Sigma_{j=1}^n v_j t_j : t \in D\}$ for each $v \in V$. Setting $s = -v$, we have $s \in R_+^n$ (because $S \subset R_-^n$). By Proposition 4, the latter subproblem is equivalent to

$$-\min_{x \in R^m} \sum_{j=1}^{n} s_j [h_j(x) - t_j^\circ] .$$

Denote by $R(s)$ the subproblem

$$\min_{x \in R^m} \sum_{j=1}^{n} s_j h_j(x) \tag{$R(s)$}$$

and let $\mu(s)$ be the optimal value, and $x(s)$ an optimal solution of $R(s)$. It is convenient to use the notation $\langle \cdot , \cdot \rangle$ for inner product.

COROLLARY 1. *If $\langle t^0, s \rangle - \mu(s) \leq 1$ for all $s \in -V$ then $\bar{t}$ is a global optimal solution of $(\tilde{Q})$.*

Suppose now that, for some $\tilde{s} \in -V$, we have

$$\langle t^\circ, \tilde{s} \rangle - \mu(\tilde{s}) > 1 . \tag{3}$$

Let $\tilde{x} = x(\tilde{s})$ and $\tilde{t} = h(\tilde{x})$ (so $\tilde{t} \in \tilde{D}$ and $\langle \tilde{t}, \tilde{s} \rangle = \mu(\tilde{s})$). If $f(\tilde{t}) > f(\bar{t})$ then we have obtained a better feasible solution than the current best $\bar{t}$. Otherwise, $f(\tilde{t}) \leq f(\bar{t})$, compute

$$\theta = \sup\{\lambda :\ f(t^\circ + \lambda(\tilde{t} - t^\circ)) \leq \gamma\} . \tag{4}$$

Note that $\theta \geq 1$ because $f(t^\circ) < \gamma$ while $f(\tilde{t}) \leq \gamma$.

PROPOSITION 6. *The cut*

$$\langle \tilde{t} - t^\circ, t \rangle \leq \frac{1}{\theta} \tag{5}$$

excludes $\tilde{v} = -\tilde{s}$ from S without excluding any point of C^*.

Proof. Since $\langle \tilde{t}, \tilde{s} \rangle = \mu(\tilde{s})$ it follows from (3) that $\langle \tilde{t} - t^0, -\tilde{s} \rangle = \langle t^\circ, \tilde{s} \rangle - \mu(\tilde{s}) > 1$; i.e., $-\tilde{s}$ violates the inequality (5). On the other hand, since from the definition of θ (see (4)), $\theta(t - t^\circ) \in C$, it follows that any point $t \in C^*$ satisfies (5). □

Thus, the polytope S', formed by applying the cut (5) to S, given by

$$S' = S \cap \left\{ t \colon \langle \tilde{t} - t^{\circ}, t \rangle \leq \frac{1}{\theta} \right\},$$

does not contain $-\tilde{s}$ but still contains C^*. Consequently, the procedure can be repeated from S' in place of S.

To start, we need a polytope $S_1 \supset C^*$. This polytope can be constructed, for example, as follows.

Let $e = (1, \ldots, 1)$ be a vector of ones in R^n. Since $0 \in \text{int}C$ one can always select $\alpha > 0$ small enough such that $-\alpha e \in C$.

PROPOSITION 7. $C^* \subset S_1 = \{t \in R^n_- : -\Sigma^n_{j=1} t_j \leq \frac{1}{\alpha}\}$.

Proof. We have $(C^*)^* = C$ (see [6]). Hence $-\alpha e \in C$ implies $\langle -\alpha e, t \rangle \leq 1$ for all $t \in C^*$; i.e., $-\alpha\, \Sigma^n_{j=1} t_j \leq 1$ for all $t \in C^*$. □

Thus, starting with S_1, we can build a sequence of nested polytopes in R^n_-

$$S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots \supset C^*$$

such that, for each k, either $S_k \subset D^*$ (then the current best $\bar{t}$ is a global optimum), or at least one $s^k \in -S_k$ has $\langle t^{\circ}, s^k \rangle - \mu(s^k) > 1$. It may be that $t^k = h(x(s^k))$ is better than $\bar{t}$ (subproblem $(Q, \bar{t})$ has now been solved and we can proceed to $(\tilde{Q}, \bar{t}^1)$ with $\bar{t}^1$ equal to the just found t^k). Otherwise, t^k generates a cut which, adjoined to S_k, determines S_{k+1}. It turns out that if we always choose $s^k \in \text{argmax}_{s \in -V_k}\{\langle t^{\circ}, s \rangle - \mu(s)\}$ then convergence of the procedure is guaranteed.

In practice we combine $(\tilde{Q}, \bar{t}^{\circ})$, $(\tilde{Q}, \bar{t}^1), \ldots,$ into a unified process. Then we have the following:

ALGORITHM

Initialization

Let $\bar{t}$ be the best available feasible solution of problem $(\tilde{Q})$. Take a feasible solution t° such that $f(t^{\circ}) < f(\bar{t})$ (preferably a t° substantially worse than $\bar{t}$ because we want t° to lie sufficiently far from the boundary of the set $\tilde{C}$). Set $\tilde{V}_1 = V_1$ (set of vertices of initial S_1) $= \{-\frac{1}{\alpha}e^i : i = 1, \ldots, n\}$. Set $k = 1$.

Iteration $k = 1, 2, \ldots$

Step 1. For each $s \in -V_k$ solve $R(s)$ to obtain the optimal value $\mu(s)$ and optimal solution $x(s)$.

Step 2. If $\max\{\langle t^{\circ}, s \rangle - \mu(s) \colon s \in -V_k\} \leq 1$, then terminate: $\bar{t}$ is the global optimal solution of $(\tilde{Q})$.

Step 3. Select $s^k \in \text{argmax}\{\langle t^{\circ}, s \rangle - \mu(s) \colon s \in -V_k\}$. Let $x^k = x(s^k)$, $t^k = h(x^k)$.

If $f(t^k) > f(\bar{t})$ then reset $\bar{t} \leftarrow t^k$ (otherwise $\bar{t}$ is unchanged). Compute

$$\theta_k = \sup\{\lambda : f(t^\circ + \lambda(t^k - t^\circ)) \leq f(\bar{t})\}$$

and define

$$S_{k+1} = S_k \cap \left\{t : \langle t^k - t^\circ, t\rangle \leq \frac{1}{\theta_k}\right\}.$$

Step 4. Compute the vertex set V_{k+1} of S_{k+1} (knowing already the vertex set V_k of S_k). Set $\bar{V}_{k+1} = V_{k+1} \backslash V_k$. Go to iteration $k+1$.

4. Convergence

Define a function $g : R^n \to [0, \infty]$ by setting for each $v \in R^n$.

$$g(v) = \sup\{\langle v, t\rangle : t \in D\}.$$

Obviously $g(v)$ is a convex function and $D^* = \{v : g(v) \leq 1\}$.

LEMMA 1. *For every k, $t^k - t^\circ \in \partial g(-s^k)$.*

Proof. Let $v^k = -s^k$. Clearly

$$g(v^k) = \langle t^\circ, s^k\rangle - \mu(s^k) = \langle t^\circ, s^k\rangle - \langle t^k, s^k\rangle = \langle t^k - t^\circ, v^k\rangle. \tag{6}$$

Hence, $g(v) - g(v^k) = g(v) - \langle t^k - t^\circ, v^k\rangle \geq \langle v, t^k - t^\circ\rangle - \langle t^k - t^\circ, v^k\rangle = \langle t^k - t^\circ, v - v^k\rangle$. □

LEMMA 2. *Any cluster point of the sequence $v^k = -s^k$ belongs to D^*.*

Proof. Denote $l_k(v) = \langle t^k - t^\circ, v\rangle$. Now consider the set D^*, the sequence $\{v^k\}$, and the sequence of affine functions $l_k(v)$. We have (see (6))

$$l_k(v^k) = \langle t^k - t^\circ, v^k\rangle = g(v^k) > 1,$$

while

$$l_k(v) = g(v^k) + \langle t^k - t^\circ, v - v^k\rangle \leq g(v) \leq 1$$

for all $v \in D^*$. That is, each $l_k(\cdot)$ strictly separates v^k from D^*. Since the sequence $\{v^k\}$ is bounded (contained in S_1) and $t^k - t^\circ \in \partial g(v^k)$ by Lemma 1, it follows from well known results in outer approximation methods (see, e.g., [4]) that any cluster point $\hat{v}$ of $\{v^k\}$ will belong to D^*. □

Denote by $\bar{t}^k$ the current best solution in iteration k.

THEOREM. *Either the algorithm terminates after finitely many iterations yielding a global optimal solution, or it generates an infinite sequence $\{\bar{t}^k\}$ every cluster point of which is a global optimal solution.*

Proof. Suppose the algorithm is infinite. We will show that in fact any cluster point $\hat{t}$ of sequence $\{\bar{t}^k\}$ is a global optimal solution. Let $\hat{t} = \lim_{\nu\to\infty} \bar{t}^{k_\nu}$. Without loss of generality, we can assume $v^{k_\nu} \to \hat{v}$. Since, by Lemma 2, $\hat{v} \in D^*$ it follows that $g(\hat{v}) \leq 1$. Let $\tilde{C}_k = \{t: f(t) \leq f(\bar{t}^k)\}$, $C_k = \tilde{C}_k - t^\circ$. Then

$$g(v^k) = \max\{g(v): v \in V_k\} = \max\{g(v): v \in S_k\} \geq \max\{g(v): v \in C_k^*\}$$

since $S_k \supset C_k^*$ by Proposition 6. But $g(v^{k_\nu}) \leq g(\hat{v}) \leq 1$. Therefore,

$$\max\left\{g(v): v \in \bigcap_{\nu=1}^{\infty} C_{k_\nu}^*\right\} \leq 1$$

which implies that $\cap_{\nu=1}^{\infty} C_{k_\nu}^* \subset D^*$, and consequently, $\cup_{\nu=1}^{\infty} C_{k_\nu} \supset D$. Thus, $\tilde{D} \subset \cup_{\nu=1}^{\infty} \tilde{C}_{k_\nu}$. That is, for any $t \in \tilde{D}$, there exists a k_ν such that $t \in \tilde{C}_{k_\nu}$; hence $f(t) \leq f(\bar{t}^{k_\nu}) \leq f(\hat{t})$, proving the global optimality of $\hat{t}$.

5. Discussion

(i) If P_k denotes the polar of the polytope S_k then

$$P_1 \subset P_2 \subset \cdots \subset P_k \subset \cdots$$

That is, the method amounts to building a sequence of expanding polyhedra all contained in the convex set $C = \{t: f(t + t^\circ) \leq \gamma_{\text{opt}}\}$, where $\gamma_{\text{opt}} = \min f(D)$, and eventually covering all of D, whence the name *polyhedral annexation*.

Every subproblem $R(s)$ is an unconstrained convex minimization problem which can be solved by efficient standard methods. If the functions $h_j(x)$ are polyhedral (as assumed in [5]), then each $R(s)$ seeks to minimize a convex piecewise affine function over R^m. Recall that $m = 2$ for the practical problem considered in [5].

We assume that two feasible solutions t° and $\bar{t}$ are available at the beginning with $f(t^\circ) < f(\bar{t})$. This assumption is innocuous here since for any x the vector $t = h(x)$ is a feasible solution.

Each polytope S_{k+1} is obtained from its predecessor S_k by adjoining a new linear constraint. Therefore, V_{k+1} can be derived from V_k using, e.g., the procedure of either Horst *et al.* [3] or [1]. If at some iteration k the set V_k becomes too large, then it is possible to restart, with the current best solution $\bar{t}^k$ as the starting $\bar{t}$. Thus, the growth of V_k may create difficulty for this method, as for similar methods of concave minimization. Based on the reported numerical experience in [3], [1] our method should be practical for values of n up to 20 on a microcomputer.

The problems $R(s)$ need not necessarily be solved to optimality. It is possible to develop an "approximate" variant of the Algorithm where each $R(s)$ can be solved to within some accuracy (but then the output of the Algorithm is an approximate global solution).

Also, it is clear that the method can be applied to nonconvex functions h if the unconstrained global minimization of these functions can be done efficiently (indeed, the convexity of $\tilde{D}$ is immaterial).

(ii) Each problem $R(s)$ is a Fermat–Weber problem where the distance from x to district j is $s_j h_j(x)$. Thus our approach amounts to solving a sequence of Fermat–Weber problems depending upon a parameter $s \in R^n_+$ (s_j is a "weight" assigned to district j) that is gradually adjusted till global optimality.

When $m = 2$ and every $h_j(x)$, $j = 1, \ldots, n$, is given by a polyhedral norm, then $h_j(x)$ is affine over each of the cones vertexed at a_j and generated by a facet of the "unit ball" associated with this norm. Therefore, the function $\Sigma^n_{j=1} s_j h_j(x)$, for given $s \in R^n_+$, is affine over each polyhedron of the form $\cap^n_{j=1} M_j$, where M_j is a cone of the type just described in which $h_j(x)$ is affine. That is, the objective function of the subproblem $R(s)$ is a convex, piecewise affine function, with the affine pieces having domains given by the elementary polyhedra discussed in the Introduction. It follows that an optimal solution of a subproblem $R(s)$, and hence a global optimal solution of problem (P), is achieved at a vertex of an elementary polyhedron, i.e., an intersection point in accordance with the result established in [5].

6. Extensions

A more general (and more difficult) problem than the one considered above occurs when two (or more) facilities have to be constructed (located). In this case, each district j will be served by the nearest facility, so the problem is to determine the locations, say x and y, of the facilities so as to maximize

$$\sum_{j=1}^{n} q_j[\tilde{h}_j(x, y)] \text{ over } (x, y) \in R^2 \times R^2 ,$$

where

$$\tilde{h}_j(x, y) = \min\{h_j(x), h_j(y)\} \tag{7}$$

and $q_j(t)$, $h_j(x)$ are functions satisfying the same assumptions (A1), (A2) as previously. Setting $z = (x, y)$, we see that this problem is of the same type as (P), except that $\tilde{h}_j(x, y)$ is no longer convex. However, as can easily be checked, the same method as above can be applied, with the subproblems $R(s)$ now being

$$\min_{(x,y) \in R^2 \times R^2} \sum_{j=1}^{n} s_j \tilde{h}_j(x, y) . \tag{8}$$

Since $\tilde{h}_j(x, y) = [h_j(x) + h_j(y)] - \max\{h_j(x), h_j(y)\}$ is a difference of two convex functions (a d.c. function), then problem (8) is an unconstrained d.c. optimization problem, equivalent to

$$\begin{aligned} &\min\Big[u - \sum_{j=1}^{n} s_j \max\{h_j(x), h_j(y)\}\Big] \\ &\text{s.t. } \sum_{j=1}^{n} s_j(h_j(x) + h_j(y)) \leq u \\ &x, y \in R^2, \quad 0 \leq u \leq L \end{aligned} \tag{9}$$

where L is a sufficiently large positive number. The latter problem (which is a concave minimization problem over a compact convex set in $R^2 \times R^2 \times R_+$) can be solved practically by currently available algorithms (see [4]).

If $h_j(x)$ is associated with a polyhedral norm then, since $h_j(x)$ is affine on each elementary polyhedron, it follows from (7) that $\tilde{h}_j(x, y)$ is concave on each polyhedron in $R^2 \times R^2$ which is the product of two elementary polyhedra in R^2. Therefore, an optimal solution of $R(s)$ is of the form $(x(s), y(s))$ where $x(s)$ and $y(s)$ are intersection points.

A case of interest is when the location y of one facility has already been fixed. Then the problem is the same as that which would arise if an old facility already exists and we want to construct a new facility at the optimal location. Obviously, problem (P) can be considered as a special case of a problem of this form, where y is taken to be infinitely far away from the origin.

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